Chapter 4

Electrostatic Fields in Matter

Problem 4.1

$E = V/x = 500/10^{-3} = 5\times 10^5$. Table 4.1: $\alpha/4\pi\epsilon_0 = 0.66\times 10^{-30}$, so $\alpha = 4\pi(8.85\times 10^{-12})(0.66\times 10^{-30}) = 7.34\times 10^{-41}$. $p = \alpha E = ed \Rightarrow d = \alpha E/e = (7.34\times 10^{-41})(5\times 10^5)/(1.6\times 10^{-19}) = 2.29\times 10^{-16}\,\text{m}$. $d/R = (2.29\times 10^{-16})/(0.5\times 10^{-10}) = \boxed{4.6\times 10^{-6}.}$ To ionize, say $d = R$. Then $R = \alpha E/e = \alpha V/ex \Rightarrow V = Rex/\alpha = (0.5\times 10^{-10})(1.6\times 10^{-19})(10^{-3})/(7.34\times 10^{-41}) = \boxed{10^8\,\text{V}.}$

Problem 4.2

First find the field, at radius r, using Gauss' law: $\int \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q_{\text{enc}}$, or $E = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}Q_{\text{enc}}$.

$$\begin{aligned}
Q_{\text{enc}} &= \int_0^r \rho\,d\tau = \frac{4\pi q}{\pi a^3}\int_0^r e^{-2\bar{r}/a}\bar{r}^2\,d\bar{r} = \frac{4q}{a^3}\left[-\frac{a}{2}e^{-2\bar{r}/a}\left(\bar{r}^2 + a\bar{r} + \frac{a^2}{2}\right)\right]\Bigg|_0^r \\
&= -\frac{2q}{a^2}\left[e^{-2r/a}\left(r^2 + ar + \frac{a^2}{2}\right) - \frac{a^2}{2}\right] = q\left[1 - e^{-2r/a}\left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2}\right)\right].
\end{aligned}$$

[*Note:* $Q_{\text{enc}}(r\to\infty) = q$.] So the field of the electron cloud is $E_{\text{e}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\left[1 - e^{-2r/a}\left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2}\right)\right]$. The proton will be shifted from $r = 0$ to the point d where $E_{\text{e}} = E$ (the external field):

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\left[1 - e^{-2d/a}\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right)\right].$$

Expanding in powers of (d/a):

$$\begin{aligned}
e^{-2d/a} &= 1 - \left(\frac{2d}{a}\right) + \frac{1}{2}\left(\frac{2d}{a}\right)^2 - \frac{1}{3!}\left(\frac{2d}{a}\right)^3 + \cdots = 1 - 2\frac{d}{a} + 2\left(\frac{d}{a}\right)^2 - \frac{4}{3}\left(\frac{d}{a}\right)^3 + \cdots \\
1 - e^{-2d/a}\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right) &= 1 - \left(1 - 2\frac{d}{a} + 2\left(\frac{d}{a}\right)^2 - \frac{4}{3}\left(\frac{d}{a}\right)^3 + \cdots\right)\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right) \\
&= \cancel{1} - \cancel{1} - 2\cancel{\frac{d}{a}} - 2\cancel{\frac{d^2}{d^2}} + 2\cancel{\frac{d}{a}} + 4\cancel{\frac{d^2}{d^2}} + 4\cancel{\frac{d^3}{d^3}} - 2\cancel{\frac{d^2}{d^2}} - 4\cancel{\frac{d^3}{d^3}} + \frac{4}{3}\frac{d^3}{a^3} + \cdots \\
&= \frac{4}{3}\left(\frac{d}{a}\right)^3 + \text{higher order terms.}
\end{aligned}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\left(\frac{4}{3}\frac{d^3}{a^3}\right) = \frac{1}{4\pi\epsilon_0}\frac{4}{3a^3}(qd) = \frac{1}{3\pi\epsilon_0 a^3}p. \quad \boxed{\alpha = 3\pi\epsilon_0 a^3.}$$

[Not so different from the *uniform* sphere model of Ex. 4.1 (see Eq. 4.2). Note that this result predicts $\frac{1}{4\pi\epsilon_0}\alpha = \frac{3}{4}a^3 = \frac{3}{4}\left(0.5\times 10^{-10}\right)^3 = 0.09\times 10^{-30}\,\text{m}^3$, compared with an experimental value (Table 4.1) of $0.66\times 10^{-30}\,\text{m}^3$. Ironically the "classical" formula (Eq. 4.2) is slightly *closer* to the empirical value.]

Problem 4.3

$\rho(r) = Ar$. Electric field (by Gauss's Law): $\oint \mathbf{E}\cdot d\mathbf{a} = E\left(4\pi r^2\right) = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\int_0^r A\bar{r}\,4\pi\bar{r}^2\,d\bar{r}$, or $E = \frac{1}{4\pi r^2}\frac{4\pi A}{\epsilon_0}\frac{r^4}{4} = \frac{Ar^2}{4\epsilon_0}$. This "internal" field balances the external field $\mathbf{E}$ when nucleus is "off-center" an amount d: $ad^2/4\epsilon_0 = E \;\Rightarrow\; d = \sqrt{4\epsilon_0 E/A}$. So the induced dipole moment is $p = ed = 2e\sqrt{\epsilon_0/A}\sqrt{E}$. Evidently $\boxed{p \text{ is proportional to } E^{1/2}.}$

For Eq. 4.1 to hold in the weak-field limit, *E must be proportional to r*, for small r, which means that ρ must go to a constant (not zero) at the origin: $\boxed{\rho(0) \neq 0}$ (nor infinite).

Problem 4.4

Field of q: $\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}$. Induced dipole moment of atom: $\mathbf{p} = \alpha\mathbf{E} = \frac{\alpha q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}}$.

Field of this dipole, at location of q ($\theta = \pi$, in Eq. 3.103): $E = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left(\frac{2\alpha q}{4\pi\epsilon_0 r^2}\right)$ (to the right).

Force on q due to this field: $\boxed{F = 2\alpha\left(\frac{q}{4\pi\epsilon_0}\right)^2\frac{1}{r^5}}$ (attractive).

Problem 4.5

Field of $\mathbf{p}_1$ at $\mathbf{p}_2$ ($\theta = \pi/2$ in Eq. 3.103): $\mathbf{E}_1 = \frac{p_1}{4\pi\epsilon_0 r^3}\,\hat{\boldsymbol{\theta}}$ (points *down*).

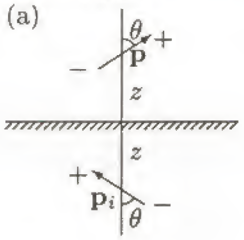
Torque on $\mathbf{p}_2$: $\mathbf{N}_2 = \mathbf{p}_2\times\mathbf{E}_1 = p_2E_1\sin 90° = p_2E_1 = \boxed{\frac{p_1p_2}{4\pi\epsilon_0 r^3}}$ (points *into* the page).

Field of $\mathbf{p}_2$ at $\mathbf{p}_1$ ($\theta = \pi$ in Eq. 3.103): $\mathbf{E}_2 = \frac{p_2}{4\pi\epsilon_0 r^3}(-2\,\hat{\mathbf{r}})$ (points to the *right*).

Torque on $\mathbf{p}_1$: $\mathbf{N}_1 = \mathbf{p}_1\times\mathbf{E}_2 = \boxed{\frac{2p_1p_2}{4\pi\epsilon_0 r^3}}$ (points *into* the page).

Problem 4.6

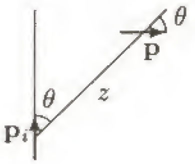
(a)



Use image dipole as shown in Fig. (a). Redraw, placing $\mathbf{p}_i$ at the origin, Fig. (b).

$$\mathbf{E}_i = \frac{p}{4\pi\epsilon_0(2z)^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}); \qquad \mathbf{p} = p\cos\theta\,\hat{\mathbf{r}} + p\sin\theta\,\hat{\boldsymbol{\theta}}.$$

(b)

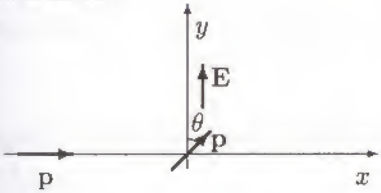


$$\begin{aligned}\mathbf{N} &= \mathbf{p}\times\mathbf{E}_i = \frac{p^2}{4\pi\epsilon_0(2z)^3}\left[(\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})\times(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})\right] \\ &= \frac{p^2}{4\pi\epsilon_0(2z)^3}\left[\cos\theta\sin\theta\,\hat{\boldsymbol{\phi}} + 2\sin\theta\cos\theta(-\,\hat{\boldsymbol{\phi}})\right] \\ &= \frac{p^2\sin\theta\cos\theta}{4\pi\epsilon_0(2z)^3}(-\hat{\boldsymbol{\phi}}) \quad \text{(out of the page).}\end{aligned}$$

But $\sin\theta\cos\theta = (1/2)\sin 2\theta$, so $\boxed{N = \dfrac{p^2 \sin 2\theta}{4\pi\epsilon_0(16z^3)}}$ (out of the page).

For $0 < \theta < \pi/2$, $\mathbf{N}$ tends to rotate $\mathbf{p}$ counterclockwise; for $\pi/2 < \theta < \pi$, $\mathbf{N}$ rotates $\mathbf{p}$ clockwise. Thus the $\boxed{\text{stable orientation is perpendicular to the surface—either } \uparrow \text{ or } \downarrow.}$

Problem 4.7



Say the field is uniform and points in the y direction. First slide $\mathbf{p}$ in from infinity along the x axis—this takes no work, since $\mathbf{F}$ is $\perp$ $d\mathbf{l}$. (If $\mathbf{E}$ is *not* uniform, slide $\mathbf{p}$ in along a trajectory $\perp$ the field.) Now rotate (counterclockwise) into final position. The torque exerted by $\mathbf{E}$ is $\mathbf{N} = \mathbf{p}\times\mathbf{E} = pE\sin\theta\,\hat{\mathbf{z}}$. The torque *we* exert is $N = pE\sin\theta$ *clockwise*, and $d\theta$ is *counterclockwise*, so the net work done by us is *negative*:

$U = \int_{\pi/2}^{\theta} pE\sin\bar\theta\, d\bar\theta = pE\left(-\cos\bar\theta\right)\Big|_{\pi/2}^{\theta} = -pE\left(\cos\theta - \cos\frac{\pi}{2}\right) = -pE\cos\theta = -\mathbf{p}\cdot\mathbf{E}.$ qed

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Problem 4.8

$U = -\mathbf{p_1}\cdot\mathbf{E_2}$, but $\mathbf{E_2} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3\left(\mathbf{p_2}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \mathbf{p_2}\right]$. So $U = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[\mathbf{p_1}\cdot\mathbf{p_2} - 3\left(\mathbf{p_1}\cdot\hat{\mathbf{r}}\right)\left(\mathbf{p_2}\cdot\hat{\mathbf{r}}\right)\right]$. qed

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Problem 4.9

(a) $\mathbf{F} = (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E}$ (Eq. 4.5); $\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\hat{\mathbf{r}} = \dfrac{q}{4\pi\epsilon_0}\dfrac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}}$.

$$\begin{aligned}
F_x &= \left(p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z}\right)\frac{q}{4\pi\epsilon_0}\frac{x}{(x^2+y^2+z^2)^{3/2}} \\
&= \frac{q}{4\pi\epsilon_0}\left\{p_x\left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}x\frac{2x}{(x^2+y^2+z^2)^{5/2}}\right] + p_y\left[-\frac{3}{2}x\frac{2y}{(x^2+y^2+z^2)^{5/2}}\right]\right. \\
&\quad \left. + p_z\left[-\frac{3}{2}x\frac{2z}{(x^2+y^2+z^2)^{5/2}}\right]\right\} = \frac{q}{4\pi\epsilon_0}\left[\frac{p_x}{r^3} - \frac{3x}{r^5}(p_x x + p_y y + p_z z)\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{p}}{r^3} - \frac{3\mathbf{r}(\mathbf{p}\cdot\mathbf{r})}{r^5}\right]_x. \\
\mathbf{F} &= \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\left[\mathbf{p} - 3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\right].}
\end{aligned}$$

(b) $\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}\left\{3\left[\mathbf{p}\cdot(-\hat{\mathbf{r}})\right](-\hat{\mathbf{r}}) - \mathbf{p}\right\} = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right]$. (This is from Eq. 3.104; the minus signs are because $\mathbf{r}$ points *toward* $\mathbf{p}$, in this problem.)

$$\mathbf{F} = q\mathbf{E} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right].}$$

[Note that the forces are equal and opposite, as you would expect from Newton's third law.]

Problem 4.10

(a) $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \boxed{kR;}$ $\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\dfrac{1}{r^3}\dfrac{\partial}{\partial r}(r^2 kr) = -\dfrac{1}{r^2}3kr^2 = \boxed{-3k.}$

(b) For $r < R$, $\mathbf{E} = \frac{1}{3\epsilon_0}\rho r\,\hat{\mathbf{r}}$ (Prob. 2.12), so $\mathbf{E} = \boxed{-(k/\epsilon_0)\,\mathbf{r}.}$

For $r > R$, same as if all charge at center; but $Q_{\text{tot}} = (kR)(4\pi R^2) + (-3k)(\frac{4}{3}\pi R^3) = 0$, so $\boxed{\mathbf{E} = 0.}$

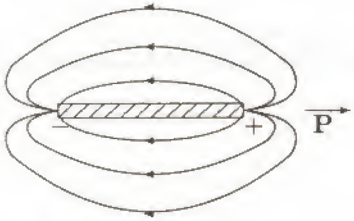
Problem 4.11

$\rho_b = 0$; $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \pm P$ (plus sign at one end—the one **P** points *toward*; minus sign at the other—the one **P** points *away* from).

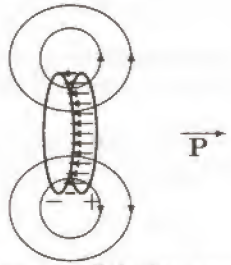
(i) $L \gg a$. Then the ends look like point charges, and the whole thing is like a physical dipole, of length L and charge $P\pi a^2$. See Fig. (a).

(ii) $L \ll a$. Then it's like a circular parallel-plate capacitor. Field is nearly uniform inside; nonuniform "fringing field" at the edges. See Fig. (b).

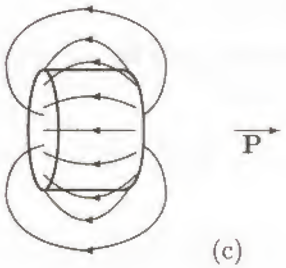
(iii) $L \approx a$. See Fig. (c).



(a) Like a dipole



(b) Like a parallel-plate capacitor



(c)

Problem 4.12

$V = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{P}\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau = \mathbf{P}\cdot\left\{\frac{1}{4\pi\epsilon_0}\int\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau\right\}$. But the term in curly brackets is precisely the *field* of a uniformly *charged* sphere, divided by ρ. The integral was done explicitly in Prob. 2.7 and 2.8:

$$\frac{1}{4\pi\epsilon_0}\int\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau = \frac{1}{\rho}\left\{\begin{array}{ll}\frac{1}{4\pi\epsilon_0}\frac{(4/3)\pi R^3\rho}{r^2}\,\hat{\mathbf{r}}, & (r > R), \\ \frac{1}{4\pi\epsilon_0}\frac{(4/3)\pi R^3\rho}{R^3}\,\mathbf{r}, & (r < R).\end{array}\right\} \quad \text{So } V(r,\theta) = \left\{\begin{array}{ll}\frac{R^3}{3\epsilon_0 r^2}\mathbf{P}\cdot\hat{\mathbf{r}} = \boxed{\frac{R^3 P\cos\theta}{3\epsilon_0 r^2},} & (r > R), \\ \frac{1}{3\epsilon_0}\mathbf{P}\cdot\mathbf{r} = \boxed{\frac{Pr\cos\theta}{3\epsilon_0},} & (r < R).\end{array}\right\}$$

Problem 4.13

Think of it as two cylinders of opposite uniform charge density $\pm\rho$. *Inside,* the field at a distance s from the axis of a uniformly charge cylinder is given by Gauss's law: $E 2\pi s\ell = \frac{1}{\epsilon_0}\rho\pi s^2\ell \Rightarrow \mathbf{E} = (\rho/2\epsilon_0)\mathbf{s}$. For *two* such cylinders, one plus and one minus, the net field (inside) is $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = (\rho/2\epsilon_0)(\mathbf{s}_+ - \mathbf{s}_-)$. But $\mathbf{s}_+ - \mathbf{s}_- = -\mathbf{d}$, so $\mathbf{E} = \boxed{-\rho\mathbf{d}/(2\epsilon_0),}$ where **d** is the vector from the negative axis to positive axis. In this case the total dipole moment of a chunk of length ℓ is $\mathbf{P}\,(\pi a^2\ell) = (\rho\pi a^2\ell)\,\mathbf{d}$. So $\rho\mathbf{d} = \mathbf{P}$, and $\boxed{\mathbf{E} = -\mathbf{P}/(2\epsilon_0),}$ for $s < a$.

Outside, Gauss's law gives $E 2\pi s\ell = \frac{1}{\epsilon_0}\rho\pi a^2\ell \Rightarrow \mathbf{E} = \frac{\rho a^2}{2\epsilon_0}\frac{\hat{\mathbf{s}}}{s}$, for *one* cylinder. For the combination, $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho a^2}{2\epsilon_0}\left(\frac{\hat{\mathbf{s}}_+}{s_+} - \frac{\hat{\mathbf{s}}_-}{s_-}\right)$, where

$$\begin{aligned}
\mathbf{s}_\pm &= \mathbf{s} \mp \frac{\mathbf{d}}{2};\\
\frac{\mathbf{s}_\pm}{s_\pm^2} &= \left(\mathbf{s} \mp \frac{\mathbf{d}}{2}\right)\left(s^2 + \frac{d^2}{4} \mp \mathbf{s}\cdot\mathbf{d}\right)^{-1} \cong \frac{1}{s^2}\left(\mathbf{s} \mp \frac{\mathbf{d}}{2}\right)\left(1 \mp \frac{\mathbf{s}\cdot\mathbf{d}}{s^2}\right)^{-1} \cong \frac{1}{s^2}\left(\mathbf{s} \mp \frac{\mathbf{d}}{2}\right)\left(1 \pm \frac{\mathbf{s}\cdot\mathbf{d}}{s^2}\right)\\
&= \frac{1}{s^2}\left(\mathbf{s} \pm \mathbf{s}\frac{(\mathbf{s}\cdot\mathbf{d})}{s^2} \mp \frac{\mathbf{d}}{2}\right) \quad \text{(keeping only 1st order terms in } \mathbf{d}\text{).}\\
\left(\frac{\hat{\mathbf{s}}_+}{s_+} - \frac{\hat{\mathbf{s}}_-}{s_-}\right) &= \frac{1}{s^2}\left[\left(\mathbf{s} + \mathbf{s}\frac{(\mathbf{s}\cdot\mathbf{d})}{s^2} - \frac{\mathbf{d}}{2}\right) - \left(\mathbf{s} - \mathbf{s}\frac{(\mathbf{s}\cdot\mathbf{d})}{s^2} + \frac{\mathbf{s}}{2}\right)\right] = \frac{1}{s^2}\left(2\frac{\mathbf{s}(\mathbf{s}\cdot\mathbf{d})}{s^2} - \mathbf{d}\right).
\end{aligned}$$

$$\boxed{\mathbf{E}(\mathbf{s}) = \frac{a^2}{2\epsilon_0}\frac{1}{s^2}\left[2(\mathbf{P}\cdot\hat{\mathbf{s}})\,\hat{\mathbf{s}} - \mathbf{P}\right],} \quad \text{for } s > a.$$

Problem 4.14

Total charge on the dielectric is $Q_{\text{tot}} = \oint_S \sigma_b\, da + \int_{\mathcal{V}} \rho_b\, d\tau = \oint_S \mathbf{P}\cdot d\mathbf{a} - \int_{\mathcal{V}} \boldsymbol{\nabla}\cdot\mathbf{P}\, d\tau$. But the divergence theorem says $\oint_S \mathbf{P}\cdot d\mathbf{a} = \int_{\mathcal{V}} \boldsymbol{\nabla}\cdot\mathbf{P}\, d\tau$, so $Q_{\text{enc}} = 0$. qed

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Problem 4.15

(a) $\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2}$; $\quad \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \left\{\begin{array}{ll} +\mathbf{P}\cdot\hat{\mathbf{r}} = k/b & (\text{at } r = b), \\ -\mathbf{P}\cdot\hat{\mathbf{r}} = -k/a & (\text{at } r = a). \end{array}\right\}$

Gauss's law $\Rightarrow \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{r^2}\,\hat{\mathbf{r}}$. For $r < a$, $Q_{\text{enc}} = 0$, so $\boxed{\mathbf{E} = 0.}$ For $r > b$, $Q_{\text{enc}} = 0$ (Prob. 4.14), so $\boxed{\mathbf{E} = 0.}$ For $a < r < b$, $Q_{\text{enc}} = \left(\frac{-k}{a}\right)(4\pi a^2) + \int_a^r \left(\frac{-k}{\bar{r}^2}\right) 4\pi\bar{r}^2 d\bar{r} = -4\pi k a - 4\pi k(r - a) = -4\pi k r$; so $\boxed{\mathbf{E} = -(k/\epsilon_0 r)\,\hat{\mathbf{r}}.}$

(b) $\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} = 0 \Rightarrow \mathbf{D} = 0$ everywhere. $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = 0 \Rightarrow \mathbf{E} = (-1/\epsilon_0)\mathbf{P}$, so $\boxed{\mathbf{E} = 0 \text{ (for } r < a \text{ and } r > b\text{);}}$ $\boxed{\mathbf{E} = -(k/\epsilon_0 r)\,\hat{\mathbf{r}} \text{ (for } a < r < b\text{).}}$

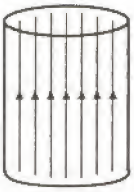
Problem 4.16

(a) Same as $\mathbf{E}_0$ minus the field at the center of a sphere with uniform polarization $\mathbf{P}$. The latter (Eq. 4.14) is $-\mathbf{P}/3\epsilon_0$. So $\boxed{\mathbf{E} = \mathbf{E}_0 + \frac{1}{3\epsilon_0}\mathbf{P}.}$ $\quad \mathbf{D} = \epsilon_0\mathbf{E} = \epsilon_0\mathbf{E}_0 + \frac{1}{3}\mathbf{P} = \mathbf{D}_0 - \mathbf{P} + \frac{1}{3}\mathbf{P}$, so $\boxed{\mathbf{D} = \mathbf{D}_0 - \frac{2}{3}\mathbf{P}.}$

(b) Same as $\mathbf{E}_0$ minus the field of ± charges at the two ends of the "needle"—but these are small, and far away, so $\boxed{\mathbf{E} = \mathbf{E}_0.}$ $\quad \mathbf{D} = \epsilon_0\mathbf{E} = \epsilon_0\mathbf{E}_0 = \mathbf{D}_0 - \mathbf{P}$, so $\boxed{\mathbf{D} = \mathbf{D}_0 - \mathbf{P}.}$

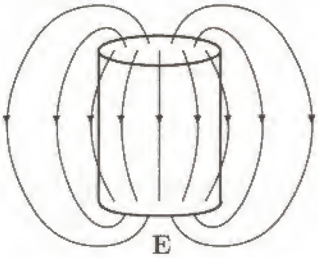
(c) Same as $\mathbf{E}_0$ minus the field of a parallel-plate capacitor with upper plate at $\sigma = P$. The latter is $-(1/\epsilon_0)P$, so $\boxed{\mathbf{E} = \mathbf{E}_0 + \frac{1}{\epsilon_0}\mathbf{P}.}$ $\quad \mathbf{D} = \epsilon_0\mathbf{E} = \epsilon_0\mathbf{E}_0 + \mathbf{P}$, so $\boxed{\mathbf{D} = \mathbf{D}_0.}$

Problem 4.17



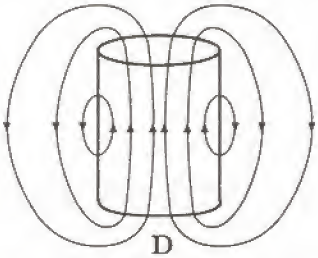
P

(uniform)



E

(field of two circular plates)

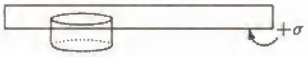


D

(same as **E** *outside*, but lines *continuous*, since $\nabla\cdot\mathbf{D}=0$)

Problem 4.18

(a) Apply $\int \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}}$ to the gaussian surface shown. $DA = \sigma A \Rightarrow \boxed{D=\sigma.}$ (*Note:* $\mathbf{D}=0$ inside the metal plate.) This is true in both slabs; **D** points *down*.



(b) $\mathbf{D}=\epsilon\mathbf{E} \Rightarrow E=\sigma/\epsilon_1$ in slab 1, $E=\sigma/\epsilon_2$ in slab 2. But $\epsilon=\epsilon_0\epsilon_r$, so $\epsilon_1=2\epsilon_0$; $\epsilon_2=\frac{3}{2}\epsilon_0$. $\boxed{E_1=\sigma/2\epsilon_0,}$ $\boxed{E_2=2\sigma/3\epsilon_0.}$

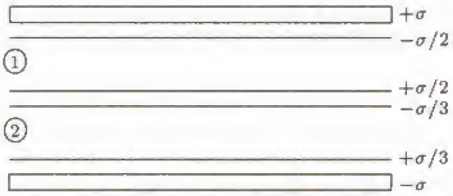
(c) $\mathbf{P}=\epsilon_0\chi_e\mathbf{E}$, so $P=\epsilon_0\chi_e d/(\epsilon_0\epsilon_r)=(\chi_e/\epsilon_r)\sigma$; $\chi_e=\epsilon_r-1\Rightarrow P=(1-\epsilon_r^{-1})\sigma.$ $\boxed{P_1=\sigma/2,}$ $\boxed{P_2=\sigma/3.}$

(d) $V=E_1a+E_2a=(\sigma a/6\epsilon_0)(3+4)=\boxed{7\sigma a/6\epsilon_0.}$

(e) $\rho_b=0;$ $\boxed{\begin{array}{l}\sigma_b=+P_1 \text{ at bottom of slab (1)} = \sigma/2,\\ \sigma_b=-P_1 \text{ at top of slab (1)} = -\sigma/2;\end{array}}$ $\boxed{\begin{array}{l}\sigma_b=+P_2 \text{ at bottom of slab (2)} = \sigma/3,\\ \sigma_b=-P_2 \text{ at top of slab (2)} = -\sigma/3.\end{array}}$

$$\text{(f) In slab 1: }\left\{\begin{array}{l}\text{total surface charge } \textit{above}\text{: } \sigma-(\sigma/2)=\sigma/2,\\ \text{total surface charge } \textit{below}\text{: } (\sigma/2)-(\sigma/3)+(\sigma/3)-\sigma=-\sigma/2,\end{array}\right\}\Longrightarrow E_1=\frac{\sigma}{2\epsilon_0}.\ \checkmark$$

$$\text{In slab 2: }\left\{\begin{array}{l}\text{total surface charge } \textit{above}\text{: } \sigma-(\sigma/2)+(\sigma/2)-(\sigma/3)=2\sigma/3,\\ \text{total surface charge } \textit{below}\text{: } (\sigma/3)-\sigma=-2\sigma/3,\end{array}\right\}\Longrightarrow E_2=\frac{2\sigma}{3\epsilon_0}.\ \checkmark$$



Problem 4.19

With no dielectric, $C_0=A\epsilon_0/d$ (Eq. 2.54).

In configuration (a), with $+\sigma$ on upper plate, $-\sigma$ on lower, $D=\sigma$ between the plates. $E=\sigma/\epsilon_0$ (in air) and $E=\sigma/\epsilon$ (in dielectric). So $V=\frac{\sigma}{\epsilon_0}\frac{d}{2}+\frac{\sigma}{\epsilon}\frac{d}{2}=\frac{Qd}{2\epsilon_0 A}\left(1+\frac{\epsilon_0}{\epsilon}\right)$.

$$C_a=\frac{Q}{V}=\frac{\epsilon_0 A}{d}\left(\frac{2}{1+1/\epsilon_r}\right)\Longrightarrow\boxed{\frac{C_a}{C_0}=\frac{2\epsilon_r}{1+\epsilon_r}.}$$

In configuration (b), with potential difference V: $E=V/d$, so $\sigma=\epsilon_0E=\epsilon_0V/d$ (in air).

$P = \epsilon_0\chi_e E = \epsilon_0\chi_e V/d$ (in dielectric), so $\sigma_b = -\epsilon_0\chi_e V/d$ (at top surface of dielectric).
$\sigma_{\text{tot}} = \epsilon_0 V/d = \sigma_f + \sigma_b = \sigma_f - \epsilon_0\chi_e V/d$, so $\sigma_f = \epsilon_0 V(1+\chi_e)/d = \epsilon_0\epsilon_r V/d$ (on top plate above dielectric).

$$\Rightarrow C_b = \frac{Q}{V} = \frac{1}{V}\left(\sigma\frac{A}{2} + \sigma_f\frac{A}{2}\right) = \frac{A}{2V}\left(\epsilon_0\frac{V}{d} + \epsilon_0\frac{V}{d}\epsilon_r\right) = \frac{A\epsilon_0}{d}\left(\frac{1+\epsilon_r}{2}\right). \boxed{\frac{C_b}{C_0} = \frac{1+\epsilon_r}{2}.}$$

[Which is greater? $\frac{C_b}{C_0} - \frac{C_a}{C_0} = \frac{1+\epsilon_r}{2} - \frac{2\epsilon_r}{1+\epsilon_r} = \frac{(1+\epsilon_r)^2-4\epsilon_r}{2(1+\epsilon_r)} = \frac{1+2\epsilon_r+4\epsilon_r^2-4\epsilon_r}{2(1+\epsilon_r)} = \frac{(1-\epsilon_r)^2}{2(1+\epsilon_r)} > 0$. So $C_b > C_a$.]
If the x axis points *down*:

	E	**D**	**P**	σ_b (top surface)	σ_f (top plate)
(a) air	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	0	0	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{V}{d}$
(a) dielectric	$\frac{2}{(\epsilon_r+1)}\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$\frac{2(\epsilon_r-1)}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$-\frac{2(\epsilon_r-1)}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}$	—
(b) air	$\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	0	0	$\frac{\epsilon_0 V}{d}$ (left)
(b) dielectric	$\frac{V}{d}\,\hat{\mathbf{x}}$	$\epsilon_r\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$(\epsilon_r-1)\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$-(\epsilon_r-1)\frac{\epsilon_0 V}{d}$	$\epsilon_r\frac{\epsilon_0 V}{d}$ (right)

Problem 4.20

$\int \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} \Rightarrow D4\pi r^2 = \rho\frac{4}{3}\pi r^3 \Rightarrow D = \frac{1}{3}\rho r \Rightarrow \mathbf{E} = (\rho r/3\epsilon)\,\hat{\mathbf{r}}$, for $r < R$; $D4\pi r^2 = \rho\frac{4}{3}\pi R^3 \Rightarrow D = \rho R^3/3r^2 \Rightarrow \mathbf{E} = (\rho R^3/3\epsilon_0 r^2)\,\hat{\mathbf{r}}$, for $r > R$.

$$V = -\int_\infty^0 \mathbf{E}\cdot d\mathbf{l} = \frac{\rho R^3}{3\epsilon_0}\frac{1}{r}\bigg|_\infty^R - \frac{\rho}{3\epsilon}\int_R^0 r\,dr = \frac{\rho R^2}{3\epsilon_0} + \frac{\rho}{3\epsilon}\frac{R^2}{2} = \boxed{\frac{\rho R^2}{3\epsilon_0}\left(1 + \frac{1}{2\epsilon_r}\right).}$$

Problem 4.21

Let Q be the charge on a length ℓ of the inner conductor.

$$\oint \mathbf{D}\cdot d\mathbf{a} = D2\pi s\ell = Q \Rightarrow D = \frac{Q}{2\pi s\ell};\quad E = \frac{Q}{2\pi\epsilon_0 s\ell}\ (a < s < b),\quad E = \frac{Q}{2\pi\epsilon s\ell}\ (b < r < c).$$

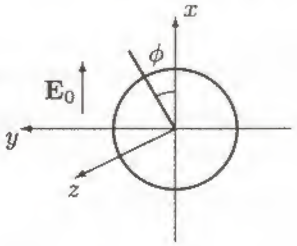
$$V = -\int_c^a \mathbf{E}\cdot d\mathbf{l} = \int_a^b \left(\frac{Q}{2\pi\epsilon_0\ell}\right)\frac{ds}{s} + \int_b^c \left(\frac{Q}{2\pi\epsilon\ell}\right)\frac{ds}{s} = \frac{Q}{2\pi\epsilon_0\ell}\left[\ln\left(\frac{b}{a}\right) + \frac{\epsilon_0}{\epsilon}\ln\left(\frac{c}{b}\right)\right].$$

$$\frac{C}{\ell} = \frac{Q}{V\ell} = \boxed{\frac{2\pi\epsilon_0}{\ln(b/a) + (1/\epsilon_r)\ln(c/b)}.}$$

Problem 4.22

Same method as Ex. 4.7: solve Laplace's equation for $V_{\text{in}}(s,\phi)$ $(s < a)$ and $V_{\text{out}}(s,\phi)$ $(s > a)$, subject to the boundary conditions

$$\begin{cases} \text{(i)} & V_{\text{in}} = V_{\text{out}} & \text{at } s = a, \\ \text{(ii)} & \epsilon\frac{\partial V_{\text{in}}}{\partial s} = \epsilon_0\frac{\partial V_{\text{out}}}{\partial s} & \text{at } s = a, \\ \text{(iii)} & V_{\text{out}} \to -E_0 s\cos\phi & \text{for } s \gg a. \end{cases}$$



From Prob. 3.23 (invoking boundary condition (iii)):

$$V_{\text{in}}(s,\phi) = \sum_{k=1}^{\infty} s^k(a_k\cos k\phi + b_k\sin k\phi),\quad V_{\text{out}}(s,\phi) = -E_0 s\cos\phi + \sum_{k=1}^{\infty} s^{-k}(c_k\cos k\phi + d_k\sin k\phi).$$

(I eliminated the constant terms by setting $V = 0$ on the $y\,z$ plane.) Condition (i) says

$$\sum a^k (a_k \cos k\phi + b_k \sin k\phi) = -E_0 s \cos\phi + \sum a^{-k} (c_k \cos k\phi + d_k \sin k\phi),$$

while (ii) says

$$\epsilon_r \sum k a^{k-1} (a_k \cos k\phi + b_k \sin k\phi) = -E_0 \cos\phi - \sum k a^{-k-1} (c_k \cos k\phi + d_k \sin k\phi).$$

Evidently $b_k = d_k = 0$ for all k, $a_k = c_k = 0$ unless $k = 1$, whereas for $k = 1$,

$$a a_1 = -E_0 a + a^{-1} c_1, \quad \epsilon_r a_1 = -E_0 - a^{-2} c_1.$$

Solving for a_1,

$$a_1 = -\frac{E_0}{(1 + \chi_e/2)}, \quad \text{so } V_{\text{in}}(s,\phi) = -\frac{E_0}{(1+\chi_e/2)} s\cos\phi = -\frac{E_0}{(1+\chi_e/2)} x,$$

and hence $\mathbf{E}_{\text{in}}(s,\phi) = -\frac{\partial V_{\text{in}}}{\partial x}\,\hat{\mathbf{x}} = \boxed{\frac{\mathbf{E}_0}{(1+\chi_e/2)}}.$ As in the spherical case (Ex. 4.7), the field inside is *uniform.*

Problem 4.23

$$\mathbf{P}_0 = \epsilon_0 \chi_e \mathbf{E}_0;\ \mathbf{E}_1 = -\frac{1}{3\epsilon_0}\mathbf{P}_0 = -\frac{\chi_e}{3}\mathbf{E}_0;\ \mathbf{P}_1 = \epsilon_0\chi_e \mathbf{E}_1 = -\frac{\epsilon_0 \chi_e^2}{3}\mathbf{E}_0;\ \mathbf{E}_2 = -\frac{1}{3\epsilon_0}\mathbf{P}_1 = \frac{\chi_e^2}{9}\mathbf{E}_0;\ \dots.$$

Evidently $\mathbf{E}_n = \left(-\frac{\chi_e}{3}\right)^n \mathbf{E}_0$, so

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 + \mathbf{E}_2 + \cdots = \left[\sum_{n=0}^{\infty} \left(-\frac{\chi_e}{3}\right)^n\right] \mathbf{E}_0.$$

The geometric series can be summed explicitly:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{so} \quad \boxed{\mathbf{E} = \frac{1}{(1+\chi_e/3)}\mathbf{E}_0,}$$

which agrees with Eq. 4.49. [Curiously, this method formally requires that $\chi_e < 3$ (else the infinite series diverges), yet the *result* is subject to no such restriction, since we can also get it by the method of Ex. 4.7.]

Problem 4.24

Potentials:

$$\begin{cases} V_{\text{out}}(r,\theta) &= -E_0 r\cos\theta + \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta), & (r > b); \\ V_{\text{med}}(r,\theta) &= \sum \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta), & (a < r < b); \\ V_{\text{in}}(r,\theta) &= 0, & (r < a). \end{cases}$$

Boundary Conditions:

$$\begin{cases} \text{(i)}\ V_{\text{out}} &= V_{\text{med}}, & (r = b); \\ \text{(ii)}\ \epsilon \frac{\partial V_{\text{med}}}{\partial r} &= \epsilon_0 \frac{\partial V_{\text{out}}}{\partial r}, & (r = b); \\ \text{(iii)}\ V_{\text{med}} &= 0, & (r = a). \end{cases}$$

$$\text{(i)} \;\Rightarrow\; -E_0 b\cos\theta + \sum \frac{B_l}{b^{l+1}} P_l(\cos\theta) = \sum \left(A_l b^l + \frac{\bar{B}_l}{b^{l+1}}\right) P_l(\cos\theta);$$

$$\text{(ii)} \;\Rightarrow\; \epsilon_r \sum \left[l A_l b^{l-1} - (l+1)\frac{\bar{B}_l}{b^{l+2}}\right] P_l(\cos\theta) = -E_0\cos\theta - \sum (l+1)\frac{B_l}{b^{l+2}} P_l(\cos\theta);$$

$$\text{(iii)} \;\Rightarrow\; A_l a^l + \frac{\bar{B}_l}{a^{l+1}} = 0 \;\Rightarrow\; \bar{B}_l = -a^{2l+1} A_l.$$

For $l \neq 1$:

$$\text{(i)} \quad \frac{B_l}{b^{l+1}} = \left(A_l b^l - \frac{a^{2l+1}A_l}{b^{l+1}}\right) \;\Rightarrow\; B_l = A_l\left(b^{2l+1} - a^{2l+1}\right);$$

$$\text{(ii)} \quad \epsilon_r\left[l A_l b^{l-1} + (l+1)\frac{a^{2l+1}A_l}{b^{l+2}}\right] = -(l+1)\frac{B_l}{b^{l+2}} \;\Rightarrow\; B_l = -\epsilon_r A_l\left[\left(\frac{l}{l+1}\right) b^{2l+1} + a^{2l+1}\right] \;\Rightarrow\; A_l = B_l = 0.$$

For $l = 1$:

$$\text{(i)} \quad -E_0 b + \frac{B_1}{b^2} = A_1 b - \frac{a^3 A_1}{b^2} \;\Rightarrow\; B_1 - E_0 b^3 = A_1 2\left(b^3 - a^3\right);$$

$$\text{(ii)} \quad \epsilon_r\left(A_1 + 2\frac{a^3 A_1}{b^3}\right) = -E_0 - 2\frac{B_1}{b^3} \;\Rightarrow\; -2B_1 - E_0 b^3 = \epsilon_r A_1\left(b^3 + 2a^3\right).$$

So $-3E_0 b^3 = A_1\left[2\left(b^3 - a^3\right) + \epsilon_r\left(b^3 + 2a^3\right)\right]$; $\quad A_1 = \dfrac{-3E_0}{2[1-(a/b)^3] + \epsilon_r[1+2(a/b)^3]}$.

$$V_{\text{med}}(r,\theta) = \frac{-3E_0}{2[1-(a/b)^3] + \epsilon_r[1+2(a/b)^3]}\left(r - \frac{a^3}{r^2}\right)\cos\theta,$$

$$\mathbf{E}(r,\theta) = -\nabla V_{\text{med}} = \boxed{\frac{3E_0}{2[1-(a/b)^3] + \epsilon_r[1+2(a/b)^3]}\left\{\left(1 + \frac{2a^3}{r^3}\right)\cos\theta\,\hat{\mathbf{r}} - \left(1 - \frac{a^3}{r^3}\right)\sin\theta\,\hat{\boldsymbol{\theta}}\right\}}.$$

Problem 4.25

There are four charges involved: (i) q, (ii) polarization charge surrounding q, (iii) surface charge (σ_b) on the top surface of the lower dielectric, (iv) surface charge (σ'_b) on the lower surface of the upper dielectric. In view of Eq. 4.39, the bound charge (ii) is $q_p = -q(\chi'_e/(1+\chi'_e)$, so the *total* (point) charge at $(0,0,d)$ is $q_t = q + q_p = q/(1+\chi'_e) = q/\epsilon'_r$. As in Ex. 4.8,

$$\text{(a) } \sigma_b = \epsilon_0\chi_e\left[\frac{-1}{4\pi\epsilon_0}\frac{qd/\epsilon'_r}{(r^2+d^2)^{\frac{3}{2}}} - \frac{\sigma_b}{2\epsilon_0} - \frac{\sigma'_b}{2\epsilon_0}\right] \quad (\text{here } \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = +P_z = \epsilon_0\chi_e E_z);$$

$$\text{(b) } \sigma'_b = \epsilon_0\chi'_e\left[\frac{1}{4\pi\epsilon_0}\frac{qd/\epsilon'_r}{(r^2+d^2)^{\frac{3}{2}}} - \frac{\sigma_b}{2\epsilon_0} - \frac{\sigma'_b}{2\epsilon_0}\right] \quad (\text{here } \sigma_b = -P_z = -\epsilon_0\chi'_e E_z).$$

Solve for σ_b, σ'_b: first divide by χ_e and χ'_e (respectively) and subtract:

$$\frac{\sigma'_b}{\chi'_e} - \frac{\sigma_b}{\chi_e} = \frac{1}{2\pi}\frac{qd/\epsilon'_r}{(r^2+d^2)^{\frac{3}{2}}} \;\Rightarrow\; \sigma'_b = \chi'_e\left[\frac{\sigma_b}{\chi_e} + \frac{1}{2\pi}\frac{qd/\epsilon'_r}{(r^2+d^2)^{\frac{3}{2}}}\right].$$

Plug this into (a) and solve for σ_b, using $\epsilon_r' = 1 + \chi_e'$:

$$\begin{aligned}
\sigma_b &= \frac{-1}{4\pi}\frac{qd/\epsilon_r'}{(r^2+d^2)^{\frac{3}{2}}}\chi_e(1+\chi_e') - \frac{\sigma_b}{2}(\chi_e+\chi_e'), \text{ so } \boxed{\sigma_b = \frac{-1}{4\pi}\frac{qd}{(r^2+d^2)^{\frac{3}{2}}}\frac{\chi_e}{[1+(\chi_e+\chi_e')/2]};} \\
\sigma_b' &= \chi_e'\left\{\frac{-1}{4\pi}\frac{qd}{(r^2+d^2)^{\frac{3}{2}}}\frac{1}{[1+(\chi_e+\chi_e')/2]} + \frac{1}{2\pi}\frac{qd/\epsilon_r'}{(r^2+d^2)^{\frac{3}{2}}}\right\}, \text{ so } \boxed{\sigma_b' = \frac{1}{4\pi}\frac{qd}{(r^2+d^2)^{\frac{3}{2}}}\frac{\epsilon_r\chi_e'/\epsilon_r'}{[1+(\chi_e+\chi_e')/2]}.}
\end{aligned}$$

The *total* bound surface charge is $\sigma_t = \sigma_b + \sigma_b' = \frac{1}{4\pi}\frac{qd}{(r^2+d^2)^{\frac{3}{2}}}\frac{(\chi_e'-\chi_e)}{\epsilon_r'[1+(\chi_e+\chi_e')/2]}$ (which vanishes, as it should, when $\chi_e' = \chi_e$). The total bound charge is (compare Eq. 4.51):

$$q_t = \frac{(\chi_e'-\chi_e)q}{2\epsilon_r'[1+(\chi_e+\chi_e')/2]} = \boxed{\left(\frac{\epsilon_r'-\epsilon_r}{\epsilon_r'+\epsilon_r}\right)\frac{q}{\epsilon_r'},} \text{ and hence}$$

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q/\epsilon_r'}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q_t}{\sqrt{x^2+y^2+(z+d)^2}}\right\}} \text{ (for } z > 0).$$

Meanwhile, since $\frac{q}{\epsilon_r'} + q_t = \frac{q}{\epsilon_r'}\left[1 + \frac{\epsilon_r'-\epsilon_r}{\epsilon_r'+\epsilon_r}\right] = \frac{2q}{\epsilon_r'+\epsilon_r}$, $\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{[2q/(\epsilon_r'+\epsilon_r)]}{\sqrt{x^2+y^2+(z-d)^2}}}$ (for $z < 0$).

Problem 4.26

From Ex. 4.5:

$$\mathbf{D} = \left\{\begin{array}{ll} 0, & (r<a) \\ \dfrac{Q}{4\pi r^2}\,\hat{\mathbf{r}}, & (r>a) \end{array}\right\}, \quad \mathbf{E} = \left\{\begin{array}{ll} 0, & (r<a) \\ \dfrac{Q}{4\pi\epsilon r^2}\,\hat{\mathbf{r}}, & (a<r<b) \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}}, & (r>b) \end{array}\right\}.$$

$$\begin{aligned}
W &= \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d\tau = \frac{1}{2}\frac{Q^2}{(4\pi)^2}4\pi\left\{\frac{1}{\epsilon}\int_a^b \frac{1}{r^2}\frac{1}{r^2}r^2\,dr + \frac{1}{\epsilon_0}\int_b^\infty \frac{1}{r^2}\,dr\right\} = \frac{Q^2}{8\pi}\left\{\frac{1}{\epsilon}\left(\frac{-1}{r}\right)\Big|_a^b + \frac{1}{\epsilon_0}\left(\frac{-1}{r}\right)\Big|_b^\infty\right\} \\
&= \frac{Q^2}{8\pi\epsilon_0}\left\{\frac{1}{(1+\chi_e)}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{b}\right\} = \boxed{\frac{Q^2}{8\pi\epsilon_0(1+\chi_e)}\left(\frac{1}{a}+\frac{\chi_e}{b}\right).}
\end{aligned}$$

Problem 4.27

Using Eq. 4.55: $W = \frac{\epsilon_0}{2}\int E^2\,d\tau$. From Ex. 4.2 and Eq. 3.103,

$$\begin{aligned}
\mathbf{E} &= \left\{\begin{array}{ll} \dfrac{-1}{3\epsilon_0}P\,\hat{\mathbf{z}}, & (r<R) \\ \dfrac{R^3P}{3\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}}), & (r>R) \end{array}\right\}, \quad \text{so} \\
W_{r<R} &= \frac{\epsilon_0}{2}\left(\frac{P}{3\epsilon_0}\right)^2\frac{4}{3}\pi R^3 = \frac{2\pi}{27}\frac{P^2R^3}{\epsilon_0}. \\
W_{r>R} &= \frac{\epsilon_0}{2}\left(\frac{R^3P}{3\epsilon_0}\right)^2\int\frac{1}{r^6}\left(4\cos^2\theta+\sin^2\theta\right)r^2\sin\theta\,dr\,d\theta\,d\phi \\
&= \frac{(R^3P)^2}{18\epsilon_0}2\pi\int_0^\pi(1+3\cos^2\theta)\sin\theta\,d\theta\int_R^\infty\frac{1}{r^4}\,dr = \frac{\pi(R^3P)^2}{9\epsilon_0}\left(-\cos\theta-\cos^3\theta\right)\Big|_0^\pi\left(-\frac{1}{3r^3}\right)\Big|_R^\infty \\
&= \frac{\pi(R^3P)^2}{9\epsilon_0}\left(\frac{4}{3R^3}\right) = \frac{4\pi R^3P^2}{27\epsilon_0}. \\
W_{\text{tot}} &= \boxed{\frac{2\pi R^3P^2}{9\epsilon_0}.}
\end{aligned}$$

This is the correct electrostatic energy of the configuration, but it is not the "total work necessary to assemble the system," because it leaves out the mechanical energy involved in polarizing the molecules.

Using Eq. 4.58: $W = \frac{1}{2}\int\mathbf{D}\cdot\mathbf{E}\,d\tau$. For $r<R$, $\mathbf{D}=\epsilon_0\mathbf{E}$, so this contribution is the same as before. For $r<R$, $\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P} = -\frac{1}{3}\mathbf{P}+\mathbf{P} = \frac{2}{3}\mathbf{P} = -2\epsilon_0\mathbf{E}$, so $\frac{1}{2}\mathbf{D}\cdot\mathbf{E} = -2\frac{\epsilon_0}{2}E^2$, and this contribution is now $(-2)\left(\frac{2\pi}{27}\frac{P^2R^3}{\epsilon_0}\right) = -\frac{4\pi}{27}\frac{R^3P^2}{\epsilon_0}$, exactly cancelling the exterior term. *Conclusion:* $\boxed{W_{\text{tot}}=0.}$ This is not surprising, since the derivation in Sect. 4.4.3 calculates the work done on the *free* charge, and in this problem there is no free charge in sight. Since this is a nonlinear dielectric, however, the result cannot be interpreted as the "work necessary to assemble the configuration"—the latter would depend entirely on *how* you assemble it.

Problem 4.28

First find the capacitance, as a function of h:

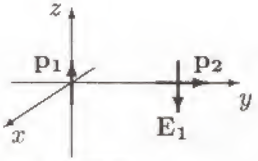
$$\left.\begin{array}{l} \textit{Air part: } E = \frac{2\lambda}{4\pi\epsilon_0 s} \Longrightarrow V = \frac{2\lambda}{4\pi\epsilon_0}\ln(b/a), \\[1em] \textit{Oil part: } D = \frac{2\lambda'}{4\pi s} \Longrightarrow E = \frac{2\lambda'}{4\pi\epsilon s} \Longrightarrow V = \frac{2\lambda'}{4\pi\epsilon}\ln(b/a), \end{array}\right\} \Longrightarrow \frac{\lambda}{\epsilon_0} = \frac{\lambda'}{\epsilon};\ \lambda' = \frac{\epsilon}{\epsilon_0}\lambda = \epsilon_r\lambda.$$

$Q = \lambda' h + \lambda(\ell - h) = \epsilon_r\lambda h - \lambda h + \lambda\ell = \lambda[(\epsilon_r-1)h+\ell] = \lambda(\chi_e h+\ell)$, where ℓ is the total height.

$$C = \frac{Q}{V} = \frac{\lambda(\chi_e h+\ell)}{2\lambda\ln(b/a)}4\pi\epsilon_0 = 2\pi\epsilon_0\frac{(\chi_e h+\ell)}{\ln(b/a)}.$$

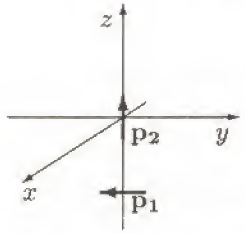
$$\left.\begin{array}{l} \text{The net upward force is given by Eq. 4.64: } F = \frac{1}{2}V^2\frac{dC}{dh} = \frac{1}{2}V^2\frac{2\pi\epsilon_0\chi_e}{\ln(b/a)}. \\ \text{The gravitational force } \textit{down} \text{ is } F = mg = \rho\pi(b^2-a^2)gh. \end{array}\right\} \boxed{h = \frac{\epsilon_0\chi_e V^2}{\rho(b^2-a^2)g\ln(b/a)}.}$$

Problem 4.29



(a) Eq. 4.5 ⇒ $\mathbf{F}_2 = (\mathbf{p}_2 \cdot \boldsymbol{\nabla})\,\mathbf{E}_1 = p_2 \dfrac{\partial}{\partial y}(\mathbf{E}_1)$;
Eq. 3.103 ⇒ $\mathbf{E}_1 = \dfrac{p_1}{4\pi\epsilon_0 r^3}\,\hat{\boldsymbol{\theta}} = -\dfrac{p_1}{4\pi\epsilon_0 y^3}\,\hat{\mathbf{z}}$. Therefore

$$\mathbf{F}_2 = -\frac{p_1 p_2}{4\pi\epsilon_0}\left[\frac{d}{dy}\left(\frac{1}{y^3}\right)\right]\hat{\mathbf{z}} = \frac{3p_1p_2}{4\pi\epsilon_0 y^4}\,\hat{\mathbf{z}}, \text{ or } \boxed{\mathbf{F}_2 = \frac{3p_1p_2}{4\pi\epsilon_0 r^4}\,\hat{\mathbf{z}}} \text{ (upward).}$$



To calculate $\mathbf{F}_1$, put $\mathbf{p}_2$ at the origin, pointing in the z direction; then $\mathbf{p}_1$ is at $-r\,\hat{\mathbf{z}}$, and it points in the $-\hat{\mathbf{y}}$ direction. So $\mathbf{F}_1 = (\mathbf{p}_1 \cdot \boldsymbol{\nabla})\,\mathbf{E}_2 = -p_1 \left.\dfrac{\partial \mathbf{E}_2}{\partial y}\right|_{x=y=0,\,z=-r}$; we need $\mathbf{E}_2$ as a function of x, y, and z.

From Eq. 3.104: $\mathbf{E}_2 = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}\left[\dfrac{3(\mathbf{p}_2\cdot\mathbf{r})\mathbf{r}}{r^2} - \mathbf{p}\right]$, where $\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$, $\mathbf{p}_2 = -p_2\,\hat{\mathbf{y}}$, and hence $\mathbf{p}_2 \cdot \mathbf{r} = -p_2 y$.

$$\begin{aligned}
\mathbf{E}_2 &= \frac{p_2}{4\pi\epsilon_0}\left[\frac{-3y(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) + (x^2+y^2+z^2)\,\hat{\mathbf{y}}}{(x^2+y^2+z^2)^{5/2}}\right] = \frac{p_2}{4\pi\epsilon_0}\left[\frac{-3xy\,\hat{\mathbf{x}} + (x^2-2y^2+z^2)\,\hat{\mathbf{y}} - 3yz\,\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{5/2}}\right]\\
\frac{\partial \mathbf{E}_2}{\partial y} &= \frac{p_2}{4\pi\epsilon_0}\left\{-\frac{5}{2}\frac{1}{r^7}2y[-3xy\,\hat{\mathbf{x}} + (x^2-2y^2+z^2)\,\hat{\mathbf{y}} - 3yz\,\hat{\mathbf{z}}] + \frac{1}{r^5}(-3x\,\hat{\mathbf{x}} - 4y\,\hat{\mathbf{y}} - 3z\,\hat{\mathbf{z}})\right\};\\
\left.\frac{\partial \mathbf{E}_2}{\partial y}\right|_{(0,0)} &= \frac{p_2}{4\pi\epsilon_0}\frac{-3z}{r^5}\,\hat{\mathbf{z}};\quad \mathbf{F}_1 = -p_1\left(\frac{p_2}{4\pi\epsilon_0}\frac{3r}{r^5}\,\hat{\mathbf{z}}\right) = \boxed{-\frac{3p_1p_2}{4\pi\epsilon_0 r^4}\,\hat{\mathbf{z}}.}
\end{aligned}$$

These results *are* consistent with Newton's third law: $\mathbf{F}_1 = -\mathbf{F}_2$.

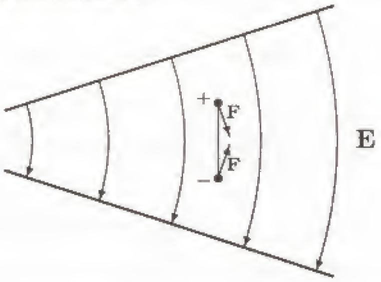
(b) From page 165, $\mathbf{N}_2 = (\mathbf{p}_2 \times \mathbf{E}_1) + (\mathbf{r} \times \mathbf{F}_2)$. The first term was calculated in Prob. 4.5; the second we get from (a), using $\mathbf{r} = r\,\hat{\mathbf{y}}$:

$$\mathbf{p}_2 \times \mathbf{E}_1 = \frac{p_1p_2}{4\pi\epsilon_0 r^3}(-\hat{\mathbf{x}});\quad \mathbf{r}\times\mathbf{F}_2 = (r\,\hat{\mathbf{y}}) \times \left(\frac{3p_1p_2}{4\pi\epsilon_0 r^4}\,\hat{\mathbf{z}}\right) = \frac{3p_1p_2}{4\pi\epsilon_0 r^3}\,\hat{\mathbf{x}};\text{ so } \boxed{\mathbf{N}_2 = \frac{2p_1p_2}{4\pi\epsilon_0 r^3}\,\hat{\mathbf{x}}.}$$

This is equal and opposite to the torque on $\mathbf{p}_1$ due to $\mathbf{p}_2$, with respect to the center of $\mathbf{p}_1$ (see Prob. 4.5).

Problem 4.30

Net force is $\boxed{\text{to the right}}$ (see diagram). Note that the field lines must bulge to the right, as shown, because $\mathbf{E}$ is perpendicular to the surface of each conductor.



Problem 4.31

$\mathbf{P} = k\mathbf{r} = k(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) \Longrightarrow \rho_b = -\nabla\cdot\mathbf{P} = -k(1+1+1) = \boxed{-3k.}$

Total volume bound charge: $\boxed{Q_{\text{vol}} = -3ka^3.}$

$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$. At top surface, $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, $z = a/2$; so $\sigma_b = ka/2$. Clearly, $\boxed{\sigma_b = ka/2}$ on all six surfaces.

Total surface bound charge: $\boxed{Q_{\text{surf}} = 6(ka/2)a^2 = 3ka^3.}$ Total bound charge is zero. ✓

Problem 4.32

$\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} \Rightarrow \mathbf{D} = \frac{q}{4\pi r^2}\,\hat{\mathbf{r}}$; $\mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \boxed{\frac{q}{4\pi\epsilon_0(1+\chi_e)}\frac{\hat{\mathbf{r}}}{r^2};}$ $\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \boxed{\frac{q\chi_e}{4\pi(1+\chi_e)}\frac{\hat{\mathbf{r}}}{r^2}.}$

$\rho_b = -\nabla\cdot\mathbf{P} = -\frac{q\chi_e}{4\pi(1+\chi_e)}\left(\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2}\right) = \boxed{-q\frac{\chi_e}{1+\chi_e}\delta^3(\mathbf{r})}$ (Eq. 1.99); $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{r}} = \boxed{\frac{q\chi_e}{4\pi(1+\chi_e)R^2};}$

$Q_{\text{surf}} = \sigma_b(4\pi R^2) = \boxed{q\frac{\chi_e}{1+\chi_e}.}$ The compensating negative charge is at the center:

$$\int \rho_b\,d\tau = -\frac{q\chi_e}{1+\chi_e}\int \delta^3(\mathbf{r})\,d\tau = -q\frac{\chi_e}{1+\chi_e}.$$

Problem 4.33

$\mathbf{E}^{\parallel}$ is continuous (Eq. 4.29); $D_{\perp}$ is continuous (Eq. 4.26, with $\sigma_f = 0$). So $E_{x_1} = E_{x_2}$, $D_{y_1} = D_{y_2} \Rightarrow \epsilon_1 E_{y_1} = \epsilon_2 E_{y_2}$, and hence

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{E_{x_2}/E_{y_2}}{E_{x_1}/E_{y_1}} = \frac{E_{y_1}}{E_{y_2}} = \frac{\epsilon_2}{\epsilon_1}. \quad \text{qed}$$

If 1 is air and 2 is dielectric, $\tan\theta_2/\tan\theta_1 = \epsilon_2/\epsilon_0 > 1$, and the field lines bend *away* from the normal. This is the opposite of light rays, so a convex "lens" would *defocus* the field lines.

Problem 4.34

In view of Eq. 4.39, the *net* dipole moment at the center is $\mathbf{p}' = \mathbf{p} - \frac{\chi_e}{1+\chi_e}\mathbf{p} = \frac{1}{1+\chi_e}\mathbf{p} = \frac{1}{\epsilon_r}\mathbf{p}$. We want the potential produced by $\mathbf{p}'$ (at the center) and σ_b (at R). Use separation of variables:

$$\left\{\begin{array}{lll} \textit{Outside:} & V(r,\theta) = \displaystyle\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta) & \text{(Eq. 3.72)} \\ \textit{Inside:} & V(r,\theta) = \displaystyle\frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{\epsilon_r r^2} + \sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta) & \text{(Eqs. 3.66, 3.102)} \end{array}\right\}.$$

$$V \text{ continuous at } R \Rightarrow \left\{\begin{array}{lll} \displaystyle\frac{B_l}{R^{l+1}} = A_l R^l, & \text{or} & B_l = R^{2l+1}A_l \ (l \neq 1) \\ \displaystyle\frac{B_1}{R^2} = \frac{1}{4\pi\epsilon_0}\frac{p}{\epsilon_r R^2} + A_1 R, & \text{or} & B_1 = \frac{p}{4\pi\epsilon_0\epsilon_r} + A_1 R^3 \end{array}\right\}.$$

$$\begin{aligned} \left.\frac{\partial V}{\partial r}\right|_{R+} - \left.\frac{\partial V}{\partial r}\right|_{R-} &= -\sum(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta) + \frac{1}{4\pi\epsilon_0}\frac{2p\cos\theta}{\epsilon_r R^3} - \sum l A_l R^{l-1}P_l(\cos\theta) = -\frac{1}{\epsilon_0}\sigma_b \\ &= -\frac{1}{\epsilon_0}\mathbf{P}\cdot\hat{\mathbf{r}} = -\frac{1}{\epsilon_0}(\epsilon_0\chi_e\mathbf{E}\cdot\hat{\mathbf{r}}) = \chi_e\left.\frac{\partial V}{\partial r}\right|_{R-} = \chi_e\left\{-\frac{1}{4\pi\epsilon_0}\frac{2p\cos\theta}{\epsilon_r R^3} + \sum l A_l R^{l-1}P_l(\cos\theta)\right\}. \end{aligned}$$

$$-(l+1)\frac{B_l}{R^{l+2}} - lA_lR^{l-1} = \chi_e l A_l R^{l-1} \ (l \neq 1);\ \text{or}\ -(2l+1)A_lR^{l-1} = \chi_e l A_l R^{l-1} \ \Rightarrow A_l = 0 \ (\ell \neq 1).$$

For $l = 1$: $-2\frac{B_1}{R^3} + \frac{1}{4\pi\epsilon_0}\frac{2p}{\epsilon_r R^3} - A_1 = \chi_e\left(-\frac{1}{4\pi\epsilon_0}\frac{2p}{\epsilon_r R^3} + A_1\right) - B_1 + \frac{p}{4\pi\epsilon_0\epsilon_r} - \frac{A_1R^3}{2} = -\frac{1}{4\pi\epsilon_0}\frac{\chi_e p}{\epsilon_r} + \chi_e\frac{A_1R^3}{2}$;

$$-\frac{p}{4\pi\epsilon_0\epsilon_r} - A_1R^3 + \frac{p}{4\pi\epsilon_0\epsilon_r} - \frac{A_1R^3}{2} = -\frac{1}{4\pi\epsilon_0}\frac{\chi_e p}{\epsilon_r} + \chi_e\frac{A_1R^3}{2} \ \Rightarrow \ \frac{A_1R^3}{2}(3+\chi_e) = \frac{1}{4\pi\epsilon_0}\frac{\chi_e p}{\epsilon_r}.$$

$$\Rightarrow A_1 = \frac{1}{4\pi\epsilon_0}\frac{2\chi_e p}{R^3\epsilon_r(3+\chi_e)} = \frac{1}{4\pi\epsilon_0}\frac{2(\epsilon_r-1)p}{R^3\epsilon_r(\epsilon_r+2)};\quad B_1 = \frac{p}{4\pi\epsilon_0\epsilon_r}\left[1 + \frac{2(\epsilon_r-1)}{(\epsilon_r+2)}\right] = \frac{p}{4\pi\epsilon_0\epsilon_r}\frac{3\epsilon_r}{\epsilon_r+2}.$$

$$\boxed{V(r,\theta) = \left(\frac{p\cos\theta}{4\pi\epsilon_0 r^2}\right)\left(\frac{3}{\epsilon_r+2}\right)(r \geq R).}$$

Meanwhile, for $r \leq R$, $V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{\epsilon_r r^2} + \frac{1}{4\pi\epsilon_0}\frac{pr\cos\theta}{R^3}\frac{2(\epsilon_r-1)}{\epsilon_r(\epsilon_r+2)}$

$$= \boxed{\frac{p\cos\theta}{4\pi\epsilon_0 r^2\epsilon_r}\left[1 + 2\left(\frac{\epsilon_r-1}{\epsilon_r+2}\right)\frac{r^3}{R^3}\right] \ (r \leq R).}$$

Problem 4.35

Given two solutions, V_1 (and $\mathbf{E}_1 = -\boldsymbol{\nabla}V_1$, $\mathbf{D}_1 = \epsilon\mathbf{E}_1$) and V_2 ($\mathbf{E}_2 = -\boldsymbol{\nabla}V_2$, $\mathbf{D}_2 = \epsilon\mathbf{E}_2$), define $V_3 \equiv V_2 - V_1$ ($\mathbf{E}_3 = \mathbf{E}_2 - \mathbf{E}_1$, $\mathbf{D}_3 = \mathbf{D}_2 - \mathbf{D}_1$).

$\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot(V_3\mathbf{D}_3)\,d\tau = \int_{\mathcal{S}} V_3\mathbf{D}_3\cdot d\mathbf{a} = 0$, ($V_3 = 0$ on $\mathcal{S}$), so $\int(\boldsymbol{\nabla}V_3)\cdot\mathbf{D}_3\,d\tau + \int V_3(\boldsymbol{\nabla}\cdot\mathbf{D}_3)\,d\tau = 0$.

But $\boldsymbol{\nabla}\cdot\mathbf{D}_3 = \boldsymbol{\nabla}\cdot\mathbf{D}_2 - \boldsymbol{\nabla}\cdot\mathbf{D}_1 = \rho_f - \rho_f = 0$, and $\boldsymbol{\nabla}V_3 = \boldsymbol{\nabla}V_2 - \boldsymbol{\nabla}V_1 = -\mathbf{E}_2 + \mathbf{E}_1 = -\mathbf{E}_3$, so $\int \mathbf{E}_3\cdot\mathbf{D}_3\,d\tau = 0$. But $\mathbf{D}_3 = \mathbf{D}_2 - \mathbf{D}_1 = \epsilon\mathbf{E}_2 - \epsilon\mathbf{E}_1 = \epsilon\mathbf{E}_3$, so $\int \epsilon(E_3)^2\,d\tau = 0$. But $\epsilon > 0$, so $\mathbf{E}_3 = 0$, so $V_2 - V_1 =$ constant. But at surface, $V_2 = V_1$, so $V_2 = V_1$ everywhere. qed

Problem 4.36

(a) Proposed potential: $\boxed{V(r) = V_0\frac{R}{r}.}$ If so, then $\boxed{\mathbf{E} = -\boldsymbol{\nabla}V = V_0\frac{R}{r^2}\,\hat{\mathbf{r}},}$ in which case $\boxed{\mathbf{P} = \epsilon_0\chi_e V_0\frac{R}{r^2}\,\hat{\mathbf{r}},}$ in the region $z < 0$. ($\mathbf{P} = 0$ for $z > 0$, of course.) Then $\sigma_b = \epsilon_0\chi_e V_0\frac{R}{R^2}(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}) = \boxed{-\frac{\epsilon_0\chi_e V_0}{R}.}$ (*Note:* $\hat{\mathbf{n}}$ points *out* of dielectric $\Rightarrow \hat{\mathbf{n}} = -\hat{\mathbf{r}}$.) This σ_b is on the surface at $r = R$. The flat surface $z = 0$ carries no bound charge, since $\hat{\mathbf{n}} = \hat{\mathbf{z}} \perp \hat{\mathbf{r}}$. Nor is there any volume bound charge (Eq. 4.39). If V is to have the required spherical symmetry, the *net* charge must be uniform:

$\sigma_{\text{tot}}4\pi R^2 = Q_{\text{tot}} = 4\pi\epsilon_0 RV_0$ (since $V_0 = Q_{\text{tot}}/4\pi\epsilon_0 R$), so $\sigma_{\text{tot}} = \epsilon_0 V_0/R$. Therefore

$$\boxed{\sigma_f = \left\{\begin{array}{ll}(\epsilon_0V_0/R), & \text{on northern hemisphere}\\ (\epsilon_0V_0/R)(1+\chi_e), & \text{on southern hemisphere}\end{array}\right\}.}$$

(b) By construction, $\sigma_{\text{tot}} = \sigma_b + \sigma_f = \epsilon_0V_0/R$ is uniform (on the northern hemisphere $\sigma_b = 0$, $\sigma_f = \epsilon_0V_0/R$; on the southern hemisphere $\sigma_b = -\epsilon_0\chi_eV_0/R$, so $\sigma_f = \epsilon V_0/R$). The potential of a uniformly charged sphere is

$$V_0 = \frac{Q_{\text{tot}}}{4\pi\epsilon_0 r} = \frac{\sigma_{\text{tot}}(4\pi R^2)}{4\pi\epsilon_0 r} = \frac{\epsilon_0V_0}{R}\frac{R^2}{\epsilon_0 r} = V_0\frac{R}{r}. \quad ✓$$

(c) Since everything is consistent, and the boundary conditions ($V = V_0$ at $r = R$, $V \to 0$ at ∞) are met, Prob. 4.35 guarantees that this is *the* solution.

(d) Figure (b) works the same way, but Fig. (a) does *not:* on the flat surface, **P** is *not* perpendicular to $\hat{\mathbf{n}}$, so we'd get bound charge on this surface, spoiling the symmetry.

Problem 4.37

$\mathbf{E}_{\text{ext}} = \dfrac{\lambda}{2\pi\epsilon_0 s}\,\hat{\mathbf{s}}$. Since the sphere is tiny, this is essentially constant, and hence $\mathbf{P} = \dfrac{\epsilon_0\chi_e}{1+\chi_e/3}\mathbf{E}_{\text{ext}}$ (Ex. 4.7).

$$\begin{aligned}\mathbf{F} &= \int \left(\frac{\epsilon_0\chi_e}{1+\chi_e/3}\right)\left(\frac{\lambda}{2\pi\epsilon_0 s}\right)\frac{d}{ds}\left(\frac{\lambda}{2\pi\epsilon_0 s}\right)\hat{\mathbf{s}}\,d\tau = \left(\frac{\epsilon_0\chi_e}{1+\chi_e/3}\right)\left(\frac{\lambda}{2\pi\epsilon_0}\right)^2\left(\frac{1}{s}\right)\left(\frac{-1}{s^2}\right)\hat{\mathbf{s}}\int d\tau \\ &= \frac{-\chi_e}{1+\chi_e/3}\left(\frac{\lambda^2}{4\pi^2\epsilon_0}\right)\frac{1}{s^3}\frac{4}{3}\pi R^3\,\hat{\mathbf{s}} = \boxed{-\left(\frac{\chi_e}{3+\chi_e}\right)\frac{\lambda^2 R^3}{\pi\epsilon_0 s^3}\,\hat{\mathbf{s}}.}\end{aligned}$$

Problem 4.38

The density of atoms is $N = \frac{1}{(4/3)\pi R^3}$. The macroscopic field **E** is $\mathbf{E}_{\text{self}} + \mathbf{E}_{\text{else}}$, where $\mathbf{E}_{\text{self}}$ is the average field over the sphere due to the atom itself.

$$\mathbf{p} = \alpha\mathbf{E}_{\text{else}} \;\Rightarrow\; \mathbf{P} = N\alpha\mathbf{E}_{\text{else}}.$$

[Actually, it is the field at the *center*, not the average over the sphere, that belongs here, but the two are in fact equal, as we found in Prob. 3.41d.] Now

$$\mathbf{E}_{\text{self}} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}$$

(Eq. 3.105), so

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\frac{\alpha}{R^3}\mathbf{E}_{\text{else}} + \mathbf{E}_{\text{else}} = \left(1 - \frac{\alpha}{4\pi\epsilon_0 R^3}\right)\mathbf{E}_{\text{else}} = \left(1 - \frac{N\alpha}{3\epsilon_0}\right)\mathbf{E}_{\text{else}}.$$

So

$$\mathbf{P} = \frac{N\alpha}{(1 - N\alpha/3\epsilon_0)}\mathbf{E} = \epsilon_0\chi_e\mathbf{E},$$

and hence

$$\chi_e = \frac{N\alpha/\epsilon_0}{(1 - N\alpha/3\epsilon_0)}.$$

Solving for α:

$$\chi_e - \frac{N\alpha}{3\epsilon_0}\chi_e = \frac{N\alpha}{\epsilon_0} \;\Rightarrow\; \frac{N\alpha}{\epsilon_0}\left(1 + \frac{\chi_e}{3}\right) = \chi_e,$$

or

$$\alpha = \frac{\epsilon_0}{N}\frac{\chi_e}{(1+\chi_e/3)} = \frac{3\epsilon_0}{N}\frac{\chi_e}{(3+\chi_e}. \quad \text{But } \chi_e = \epsilon_r - 1, \text{ so } \alpha = \frac{3\epsilon_0}{N}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right). \quad \text{qed}$$

Problem 4.39

For an ideal gas, N = Avagadro's number/22.4 liters $= (6.02 \times 10^{23})/(22.4 \times 10^{-3}) = 2.7 \times 10^{25}$. $N\alpha/\epsilon_0 = (2.7 \times 10^{25})(4\pi\epsilon_0 \times 10^{-30})\beta/\epsilon_0 = 3.4 \times 10^{-4}\beta$, where β is the number listed in Table 4.1.

$$\left.\begin{array}{lll}\text{H:} & \beta = 0.667, & N\alpha/\epsilon_0 = (3.4 \times 10^{-4})(0.67) = 2.3 \times 10^{-4}, \quad \chi_e = 2.5 \times 10^{-4} \\ \text{He:} & \beta = 0.205, & N\alpha/\epsilon_0 = (3.4 \times 10^{-4})(0.21) = 7.1 \times 10^{-5}, \quad \chi_e = 6.5 \times 10^{-5} \\ \text{Ne:} & \beta = 0.396, & N\alpha/\epsilon_0 = (3.4 \times 10^{-4})(0.40) = 1.4 \times 10^{-4}, \quad \chi_e = 1.3 \times 10^{-4} \\ \text{Ar:} & \beta = 1.64, & N\alpha/\epsilon_0 = (3.4 \times 10^{-4})(1.64) = 5.6 \times 10^{-4}, \quad \chi_e = 5.2 \times 10^{-4}\end{array}\right\}\text{ agreement is quite good.}$$

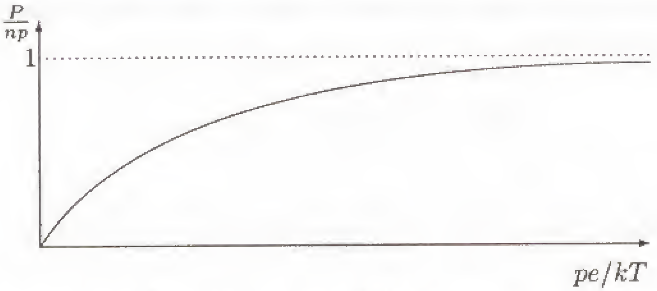
Problem 4.40

(a)

$$\begin{aligned}\langle u\rangle &= \frac{\int_{-pE}^{pE} u e^{-u/kT}\,du}{\int_{-pE}^{pE} e^{-u/kT}\,du} = \frac{(kT)^2 e^{-u/kT}\left[-(u/kT)-1\right]\big|_{-pE}^{pE}}{-kTe^{-u/kT}\big|_{-pE}^{pE}}\\ &= kT\left\{\frac{\left[e^{-pE/kT}-e^{pE/kT}\right]+\left[(pE/kT)e^{-pE/kT}+(pE/kT)e^{pE/kT}\right]}{e^{-pE/kT}-e^{pE/kT}}\right\}\\ &= kT - pE\left[\frac{e^{pE/kT}+e^{-pE/kT}}{e^{pE/kT}-e^{-pE/kT}}\right] = kT - pE\coth\left(\frac{pE}{kT}\right).\end{aligned}$$

$$\mathbf{P} = N\langle\mathbf{p}\rangle;\ \mathbf{p} = \langle p\cos\theta\rangle\hat{\mathbf{E}} = \langle\mathbf{P}\cdot\mathbf{E}\rangle(\hat{\mathbf{E}}/E) = -\langle u\rangle(\hat{\mathbf{E}}/E);\ P = Np\frac{-\langle u\rangle}{pE} = \boxed{Np\left\{\coth\left(\frac{pE}{kT}\right)-\frac{kT}{pE}\right\}}.$$

Let $y \equiv P/Np$, $x \equiv pE/kT$. Then $y = \coth x - 1/x$. As $x \to 0$, $y = \left(\frac{1}{x}+\frac{x}{3}-\frac{x^3}{45}+\cdots\right)-\frac{1}{x} = \frac{x}{3}-\frac{x^3}{45}+\cdots \to 0$, so the graph starts at the origin, with an initial slope of $1/3$. As $x\to\infty$, $y \to \coth(\infty) = 1$, so the graph goes asymptotically to $y = 1$ (see Figure).



(b) For small x, $y \approx \frac{1}{3}x$, so $\frac{P}{Np} \approx \frac{pE}{3kT}$, or $P \approx \frac{Np^2}{3kT}E = \epsilon_0\chi_e E \Rightarrow P$ is proportional to E, and $\boxed{\chi_e = \frac{Np^2}{3\epsilon_0 kT}}$.

For water at $20° = 293\,\mathrm{K}$, $p = 6.1\times10^{-30}\,\mathrm{C\,m}$; $N = \frac{\text{molecules}}{\text{volume}} = \frac{\text{molecules}}{\text{mole}}\times\frac{\text{moles}}{\text{gram}}\times\frac{\text{grams}}{\text{volume}}$.
$N = (6.0\times10^{23})\times\left(\frac{1}{18}\right)\times(10^6) = 0.33\times10^{29}$; $\chi_e = \frac{(0.33\times10^{29})(6.1\times10^{-30})^2}{(3)(8.85\times10^{-12})(1.38\times10^{-23})(293)} = \boxed{12.}$ Table 4.2 gives an experimental value of 79, so it's pretty far off.

For water vapor at $100° = 373\,\mathrm{K}$, treated as an ideal gas, $\frac{\text{volume}}{\text{mole}} = (22.4\times10^{-3})\times\left(\frac{373}{293}\right) = 2.85\times10^{-2}\ \mathrm{m}^3$.

$$N = \frac{6.0\times10^{23}}{2.85\times10^{-2}} = 2.11\times10^{25};\quad \chi_e = \frac{(2.11\times10^{25})(6.1\times10^{-30})^2}{(3)(8.85\times10^{-12})(1.38\times10^{-23})(373)} = \boxed{5.7\times10^{-3}.}$$

Table 4.2 gives 5.9×10^{-3}, so this time the agreement is quite good.
